PHYSICAL REVIEW E **74**, 046210 (2006)

Validity of the essential assumption in a projection operator method

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(Received 21 April 2006; revised manuscript received 18 August 2006; published 13 October 2006)

The projection operator method developed by Mori involves the essential assumption that chaotic motion is successfully divided into a coherent motion and a fluctuating one. We investigate the validity of the assumption using the Kuramoto-Sivashinsky equation as a model equation of chaotic systems. It has been found that the assumption is reasonable for both long wave modes and short wave modes. We have also evaluated a value of the eddy viscosity as 9.0 by extracting the nonlinear term from the coherent part. This value is consistent with the former estimates with other methods.

DOI: 10.1103/PhysRevE.74.046210 PACS number(s): 05.45.−a, 47.52.+j, 05.40.−a

I. INTRODUCTION

Two types of fluctuation-dissipation (FD) theorems, called the FD theorem of the first kind and that of the second kind, exist. Both have played an important role in systems close to thermodynamic equilibrium [1]. The former, which is the well-known Green-Kubo formula, expresses the relation between the response function and time correlation function of a physical quantity [2]. The latter, which is derived with a projection operator method, expresses the relation between the memory function and time correlation of the fluctuating force [3].

The Green-Kubo formula has been very successful in evaluating important physical quantities, such as conductivity, in systems close to thermodynamic equilibrium, because all nonequilibrium states can be expressed using an equilibrium state, whose distribution function is already known as the canonical distribution. The formula also holds in chaotic systems, which violate thermodynamic equilibrium, close to strange attractors [4,5]. However, the formula includes an unknown distribution function, and hence it only provides qualitative information. It is interesting that the Green-Kubo formula holds approximately in some chaotic systems [4,5], and a Lagrangian direct-interaction approximation in homogeneous isotropic turbulence derives the Green-Kubo formula naturally [6,7].

Although the FD theorem of the second kind includes the fluctuating force, which is difficult to evaluate numerically, it is also valid in systems that are close to strange attractors [8]; further, its form is the same as that in systems close to the thermodynamic equilibrium. This fact, as well as the noninclusion of the distribution function, is the most important characteristic of the FD theorem. Therefore, the chaotic systems can be treated similarly to the thermodynamic equilibrium ones.

This paper investigates the validity of the essential assumption, the validity of the Markov approximation, and the numerical evaluation of the eddy viscosity in the Kuramoto-Sivashinsky (KS) equation. We explain them separately as follows.

The theoretical treatment of chaotic systems is the same as that of thermodynamic equilibrium systems in the projection operator formalism. Thus, a generalized Langevin equation is derived even in chaotic systems [8,9]. The FD theorem of the second kind is exact in chaotic systems, and furthermore is formally correct even for periodic systems. Of course, the FD theorem has no physical meaning in the periodic systems, while it is only meaningful under the essential assumption that the projection operator successfully divides the chaotic motion into a coherent motion and a fluctuating one. If the essential assumption is satisfactory, the projection operator method may be useful in understanding systems that are close to strange attractors. For example, the mean values can be correctly evaluated with the projection operator method in some cases [10,11], which suggests that the assumption is appropriate. Therefore, it is interesting to investigate the validity of the assumption.

In the projection operator formalism, an evolution equation of the time correlation function is derived with a convolution integral, incorporating the memory function. In general, the convolution integral at time t, called a memory term, depends on the history of the motion for times before t. For simplicity, the memory term is often considered to be independent of the history. This is called the Markov approximation [12]. Although the Markov approximation is useful for evaluating mean values in some cases [10,11], it is not expected to be useful for evaluating the time correlation function [13]. We investigate the validity of the Markov approximation using the KS equation.

Eddy viscosity is one of the most interesting quantities in turbulence, especially for turbulence modeling [14]. It is clear that there is some relation between the eddy viscosity and memory term regarding the coherent motion in the projection operator formalism [1]. Iwayama and Okamoto [15,16] roughly evaluated the eddy viscosity for a two-dimensional inviscid barotropic fluid with the projection operator method, while we correctly evaluate the eddy viscosity being extracted from the memory term in the framework of the projection operator method using the KS equation. The obtained value is compared with the previous estimates with other methods [17–19].

II. DERIVATION OF A GENERALIZED LANGEVIN EQUATION

We treat the KS equation [20,21]

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$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \tag{2.1}$$

under the L-periodic boundary condition, $u(x,t)=u(x+L,t)$. The spatial period L is chosen to be 500, which is sufficiently large for the KS equation to produce chaotic solutions. The N-truncated Fourier transform of (2.1) yields N time evolution equations

$$f_n(t) \equiv \frac{d\hat{u}_n(t)}{dt} = L_n\hat{u}_n(t) + N_n(t), \quad n = 1, \ldots, N, \tag{2.2}$$

where the Fourier coefficient $\hat{u}_n(t)$ of $u(x,t)$ is defined as follows:

$$\hat{u}_n(t) = \int_0^L u(x,t)e^{-ik_n x}dx,$$

and the coefficient L_n of the linear term and the nonlinear term $N_n(t)$ are expressed

$$L_n = k_n^2 - k_n^4,$$

and

$$N_n(t) = -\frac{i}{L}\sum_{j=-N}^{N} k_j\hat{u}_{n-j}(t)\hat{u}_j(t), \tag{2.3}$$

respectively. Here, the wave number k_n is defined as $k_n \equiv 2n\pi/L$.

We introduce a projection operator $\mathcal{P}$ defined as [8]

$$\mathcal{P}g(\hat{\boldsymbol{u}}(t)) = \sum_{m=-N}^{N}\sum_{n=-N}^{N}\langle g(\hat{\boldsymbol{u}}(t))\hat{u}_m^*\rangle I_{mn}\hat{u}_n, \tag{2.4}$$

where $g(\hat{\boldsymbol{u}}(t))$ is an arbitrary function of $\hat{\boldsymbol{u}}(t)$, * denotes complex conjugation, and I_{mn} is a square matrix defined as follows:

$$I_{mn} \equiv [\langle\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}^\dagger\rangle^{-1}]_{mn}. \tag{2.5}$$

Here, $\dagger$ denotes Hermite conjugation. In this paper, we employ the expression $\hat{u}_n$ in place of $\hat{u}_n(0)$ at the initial time $t=0$ for simplicity and assume that the initial point (i.e., that at $t=0$) is on the attractor; this means that the solution of (2.1) is in a statistically steady state for $t \geqslant 0$. The time correlation $\langle h(t)h\rangle$ of $h(t)$ is defined as the following time integral:

$$\langle h(t)h\rangle \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T h(t+s)h(s)ds. \tag{2.6}$$

The time average (2.6) is considered to be equivalent to the ensemble average in the statistically steady state [22].

Using the projection operator, transforming the nonlinear term $N_n(t)$, and substituting the nonlinear term into the KS equation (2.2), we obtain the generalized Langevin equation

$$\frac{d\hat{u}_n(t)}{dt} = L_n\hat{u}_n(t) + \sum_{j=-N}^{N}\Omega_{nj}\hat{u}_j(t) - \sum_{j=-N}^{N}\int_0^t \Gamma_{nj}(s)\hat{u}_j(t-s)ds + r_n(t), \tag{2.7}$$

where

$$\Omega_{nj} \equiv \sum_{l=-N}^{N}\langle N_n\hat{u}_l^*\rangle I_{lj}, \tag{2.8}$$

$$\Gamma_{nj}(t) \equiv -\sum_{l=-N}^{N}\langle[\Lambda r_n(t)]\hat{u}_l^*\rangle I_{lj} = \sum_{l=-N}^{N}\langle r_n(t)r_l^*\rangle I_{lj},$$

$$r_n(t) \equiv e^{\mathcal{Q}\Lambda t}\mathcal{Q}N_n,$$

and

$$\mathcal{Q} \equiv 1 - \mathcal{P}. \tag{2.9}$$

Here, the operator Λ is defined

$$\Lambda \equiv \sum_{n=1}^{N}(L_n\hat{u}_n + N_n)\frac{\partial}{\partial\hat{u}_n}.$$

The first term of the right-hand side of (2.7) is the same as the linear term of (2.2). The second term is the projected term $\mathcal{P}N_n(t)$ of the nonlinear term. This term exhibits coherent motion. The convolution integral of the third term is also related to coherent motion; however, this coherent motion is extracted from the unprojected term $\mathcal{Q}N_n(t)$. The third term depends on the entire history of the evolution of $\hat{u}_n(t)$ and is related to friction, including the memory effect. For this reason, it is called the memory integral or memory term. The function $\Gamma_{nj}(s)$ in the third term is called the memory function. This function is related to a type of dissipation due to chaotic mixing such as an eddy viscosity in turbulent flows [14]. The last term, $r_n(t)$, is considered to be a fluctuating force because $r_n(t)$ is related to the abnormal time evolution operator $\exp(\mathcal{Q}\Lambda t)$ of the unprojected part $\mathcal{Q}N_n$; further, its time evolution may thus be very complicated. Note that $\hat{u}_n(t)$ can be expressed

$$\hat{u}_n(t) = e^{\Lambda t}\hat{u}_n,$$

by using the normal time evolution operator $\exp(\Lambda t)$.

By using the statistical homogeneity (B1) and (B2), we obtain

$$\frac{d\hat{u}_n(t)}{dt} = (L_n + \Omega_n)\hat{u}_n(t) - \int_0^t \Gamma_n(s)\hat{u}_n(t-s)ds + r_n(t), \tag{2.10}$$

from (2.7), where

$$\Gamma_n(t) = \frac{\langle r_n(t)r_n^*\rangle}{\langle\hat{u}_n\hat{u}_n^*\rangle}.$$

Then (2.10) can be transformed into

$$\frac{d\hat{u}_n(t)}{dt} = -\int_0^t \Gamma_n(s)\hat{u}_n(t-s)ds + r_n(t), \tag{2.11}$$

using

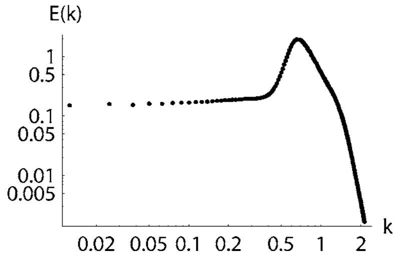

FIG. 1. Energy spectrum.

$$L_n + \Omega_n = 0. \qquad (2.12)$$

A detailed derivation of (2.12) is given in Appendix B. By multiplying (2.11) by $\hat{u}_n^*$ and then averaging, we obtain a set of evolution equations for $U_n(t)$,

$$\frac{dU_n(t)}{dt} = -\int_0^t \Gamma_n(s) U_n(t-s) ds, \qquad (2.13)$$

where $U_n(t)$ is the time correlation function in Fourier space, defined as follows:

$$U_n(t) \equiv \langle \hat{u}_n(t) \hat{u}_n^* \rangle. \qquad (2.14)$$

Further, we have used the relation $\langle r_n(t)\hat{u}_n^* \rangle = 0$, derived from $\mathcal{P}\mathcal{Q}=0$. It is important to note that (2.13) is an exact equation under the assumptions of statistical homogeneity, steadiness, and parity invariance.

The energy spectrum $E(k_n)$ is related to U_n as follows:

$$E(k_n) = \frac{U_n}{2\pi L}.$$

III. NUMERICAL RESULTS

We used a pseudospectral method with $N=256$ for the spatial derivative and the fourth-order Runge-Kutta method with a time increment of 0.1 for the time evolution. The initial value $\hat{u}_n(0)$ was set to $10^{-6}(1+i)$ for every n. The time correlation function $U_n(t)$ for $0 \leqslant t \leqslant 40$ was numerically evaluated as follows:

$$U_n(t) = \frac{1}{T-T_0}\int_{T_0}^{T} \hat{u}_n(t+s)\hat{u}_n(s) ds \qquad (3.1)$$

$$= \frac{1}{M}\sum_{j=0}^{M-1} \hat{u}_n(t+T_0+40j)\hat{u}_n(T_0+40j), \qquad (3.2)$$

where the starting time is $T_0=1000$; the final time, $T=10^7$; and the "ensemble" number, $M=(T-T_0)/40 \approx 2.5\times 10^4$. The starting time was selected to be sufficiently large, so that the data corresponding to behavior at a distance from the attractor are not included. We used (3.2) instead of (3.1) to avoid double counting the data.

Figure 1 shows the energy spectrum $E(k_n)$. We investigate the time correlation of two modes, $n=5$ ($k_n=0.06$) and $n=55$ ($k_n=0.69$), which are called the long wave and short wave modes, respectively. The latter corresponds to the dominant mode in the energy spectrum, and the characteristic length scale is $2\pi/0.69 \approx 9$. The characteristic time scale T_d is approximately evaluated as 13 from the balance between u_t and u_{xx} in (2.1).

A. Decomposition between a coherent motion and a fluctuating one

In this subsection, we investigate the validity of the essential assumption that the chaotic motion is successfully divided into a coherent motion and a fluctuating one. For this purpose, we compare two time correlation functions, $F_n(t)$ and $R_n(t)$, of the chaotic motion $f_n(t)=d\hat{u}_n(t)/dt$ and the motion $r_n(t)$ that is expected to be fluctuating, respectively. Here,

$$F_n(t) \equiv \langle f_n(t) f_n^* \rangle,$$

and

$$R_n(t) \equiv \langle r_n(t) r_n^* \rangle = U_n \Gamma_n(t). \qquad (3.3)$$

If the characteristic time scale of $R_n(t)$ is much smaller than that of $F_n(t)$, the coherent motion is successfully subtracted from the chaotic motion $f_n(t)$ by the projection operator $\mathcal{P}$; hence, the essential assumption is satisfactory.

The time correlation $F_n(t)$ can be directly evaluated with a numerical simulation, while $R_n(t)$ cannot be directly evaluated because it is difficult to evaluate $r_n(t)$. However, $R_n(t)$ can be numerically evaluated through $\Gamma_n(t)$ using (3.3) for statistically steady states. Relation (3.3) represents the FD theorem of the second kind [8]. The relation is exactly not only for a chaotic motion but also for a periodic motion. It is important to note that the relation has a meaning with respect to the FD theorem only if the essential assumption is satisfactory. A detailed treatment of the numerical evaluation of $\Gamma_n(t)$ is given in Appendix C.

For later convenience, we define the motion $s_n(t)$, which is expected to be coherent, as follows:

$$s_n(t) \equiv -\int_0^t \Gamma_n(s)\hat{u}_n(t-s) ds,$$

and then obtain the relation

$$f_n(t) = s_n(t) + r_n(t),$$

from (2.11).

1. Long wave mode ($n=5$)

We now investigate the validity of the essential assumption in the case of $n=5$ as a representative of the long wave modes using two methods: a comparison of the time correlation functions and another of time profiles.

First, we compare the two time correlation functions $F_5(t)$ and $R_5(t)$. Figure 2 shows the time correlation function $F_5(t)$ of the chaotic motion $f_5(t)$. The figure indicates that its correlation time is larger than 40. This means that the chaotic motion $f_5(t)$ includes a very slowly varying motion, as expected. The most interesting question is whether the slowly varying motion is extracted by the projection operator. Fig-

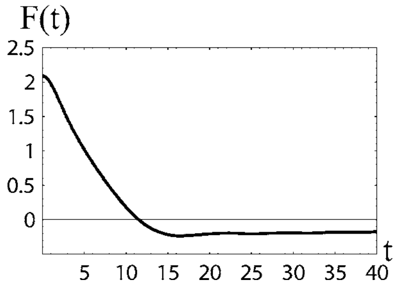

FIG. 2. Time correlation function $F_5(t)=\langle f_5(t)f_5^*\rangle$ for the long wave mode $n=5$.

ure 3 shows the time correlation function $R_5(t)$ of the motion $r_5(t)$ that is expected to be fluctuating. Because the time correlation function $R_5(t)$ is nearly zero for $t>15$, its correlation time is about 15. This is much shorter than the correlation time of $F_5(t)$. As a result of this difference in the correlation time, the slowly varying motion is extracted by the projection operator. Therefore, the chaotic motion is successfully divided into a coherent motion and a fluctuating one using the projection operator method for the long wave mode $n=5$.

Second, we compare three types of motion: the chaotic motion $f_5(t)$, the motion $s_5(t)$ that is expected to be coherent, and the motion $r_5(t)$ that is expected to be fluctuating. This comparison is more intuitive than the former one. Figure 4 shows the three types of motion $f_5(t)$, $s_5(t)$, and $r_5(t)$ for $0 \leqslant t \leqslant 500$. It is clear that the characteristic time scale of $s_5(t)$ is much larger than that of $r_5(t)$. We can strongly confirm from the figure that $s_5(t)$ is the coherent motion and $r_5(t)$ is the fluctuating motion; hence, the essential assumption is valid in the case of long wave modes. Note that the initial point (i.e., that at $t=0$) is on the attractor.

2. Short wave mode $(n=55)$

We now investigate the validity of the essential assumption in the case that $n=55$, which is a representative of the short wave modes, with the procedure employed in the case of the long wave modes.

First, we compare the two time correlation functions $F_{55}(t)$ and $R_{55}(t)$. Figure 5 shows the time correlation function $F_{55}(t)$ of the chaotic motion $f_{55}(t)$. Although $f_{55}(t)$ does not include a very slowly varying motion whose correlation time is more than 25, it includes a slowly varying motion

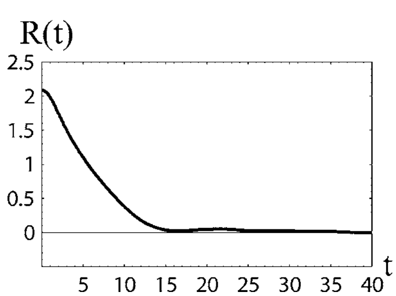

FIG. 3. Time correlation function $R_5(t)=\langle r_5(t)r_5^*\rangle$ for the long wave mode $n=5$.

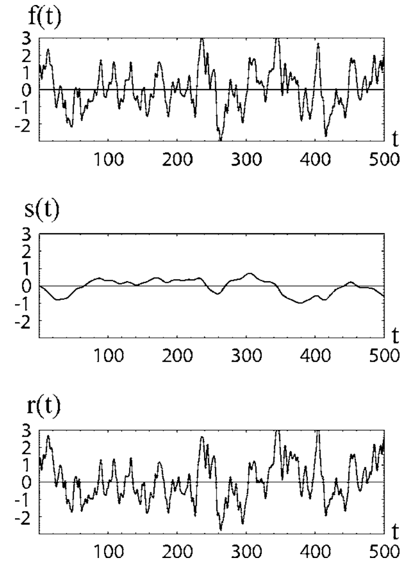

FIG. 4. The top, middle, and bottom figures show the chaotic motion $f_5(t)$, the motion $s_5(t)$ that is expected to be coherent, and the motion $r_5(t)$ that is expected to be fluctuating, respectively.

whose correlation time is about 20. Figure 6 shows the time correlation function $R_{55}(t)$ of the motion $r_{55}(t)$, which is expected to be fluctuating. The correlation time of $R_{55}(t)$ is about 15 because $R_{55}(t)$ becomes zero for $t>15$. The result of the comparison between Fig. 5 and Fig. 6 shows that the slowly varying motion whose characteristic time scale is between 15 and 25 is extracted by the projection operator. Therefore, the chaotic motion is successfully divided into a coherent motion and a fluctuating one using the projection operator method even for the short wave mode $n=55$.

Second, we compare three types of motion: the chaotic motion $f_{55}(t)$, the motion $s_{55}(t)$ that is expected to be coherent, and the motion $r_{55}(t)$ that is expected to be fluctuating. Figure 7 shows the three types of motion $f_{55}(t)$, $s_{55}(t)$, and $r_{55}(t)$ for $0 \leqslant t \leqslant 500$. We can confirm from the figure that $s_{55}(t)$ is the coherent motion and $r_{55}(t)$ is the fluctuating motion; hence, the essential assumption is also valid in the case of the long wave modes.

B. Markov approximation

In this subsection, we investigate whether the Markov approximation is valid for the two modes: the long wave mode

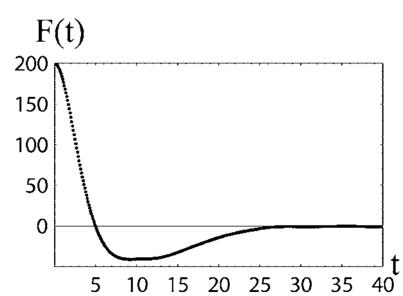

FIG. 5. Time correlation function $F_{55}(t)=\langle f_{55}(t)f_{55}^*\rangle$ for the short wave mode $n=55$.

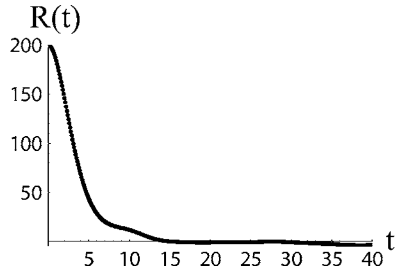

FIG. 6. Time correlation function $R_{55}(t)=\langle r_{55}(t)r_{55}^*\rangle$ for the short wave mode $n=55$.

$n=5$ and the short wave mode $n=55$. The Markov approximation is defined

$$\Gamma_n(t) = \delta(t)\int_0^\infty \Gamma_n(s)ds. \tag{3.4}$$

From (3.4), we obtain the relation

$$\int_0^t \Gamma_n(s)U_n(t-s)ds = U_n(t)\int_0^\infty \Gamma_n(s)ds,$$

which means that the memory term is independent of the history of the motion for times before t. If the characteristic time scale of the memory function $\Gamma_n(t)$ is much smaller than that of the correlation function $U_n(t)$, the Markov approximation is appropriate. Note that "Markov approximation," which is familiar in the theory of probability, has a somewhat different usage in nonequilibrium statistical mechanics [12].

Figure 8 shows the normalized time correlation function $U_5(t)/U_5$ and the normalized memory function $\Gamma_5(t)/\Gamma_5$ for the long wave mode $n=5$. The characteristic time scale of the long wave motion is larger than 40 because it includes a very slowly varying motion, while the characteristic time scale of the memory function $\Gamma_5(t)$ is about 15, which is much smaller than the characteristic time scale of the time correlation function $U_5(t)$. Therefore, the Markov approximation is satisfactory for long wave modes.

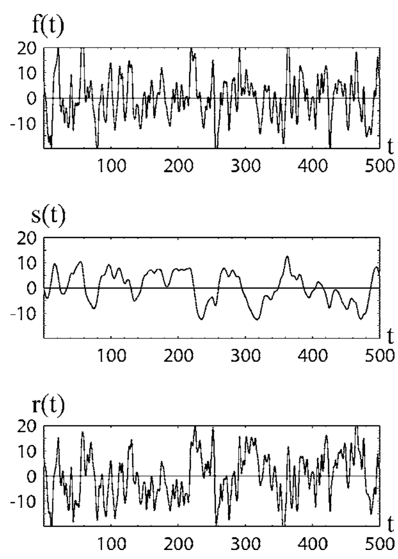

FIG. 7. The top, middle, and bottom figures show the chaotic motion $f_{55}(t)$, the motion $s_{55}(t)$ that is expected to be coherent, and the motion $r_{55}(t)$ that is expected to be fluctuating, respectively.

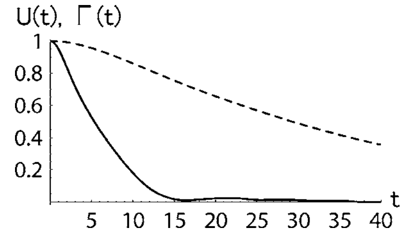

FIG. 8. Normalized time correlation functions for the long wave mode $n=5$: _ _ _, $U_5(t)/U_5$; ____, $\Gamma_5(t)/\Gamma_5$.

Figure 9 shows the normalized time correlation function $U_{55}(t)/U_{55}$ and the normalized memory function $\Gamma_{55}(t)/\Gamma_{55}$ for the short wave mode $n=55$. The characteristic time scale of the short wave motion is about 20, while the characteristic time scale of the memory function $\Gamma_{55}(t)$ is about 10, which is similar to the characteristic time scale of the time correlation function $U_{55}(t)$. Therefore, the Markov approximation is not satisfactory for short wave modes.

C. Eddy viscosity

In this subsection, we evaluate the eddy viscosity from the memory term in (2.11). The large scale properties of the KS equation are described by a nosy Burgers equation [23]

$$\tilde{u}_t = \nu_T\tilde{u}_{xx} - \tilde{u}\tilde{u}_x + \xi, \tag{3.5}$$

where $\tilde{u}$ is composed of large scale modes, ν_T is the eddy viscosity, and ξ denotes the nosy term. Note that ν_T and the common eddy viscosity have a difference of one, which is explained in Appendix A. The Fourier transform $\hat{\xi}_n(t)$ of $\xi(x,t)$ satisfies

$$\langle\hat{\xi}_n(t)\hat{\xi}_m(t')\rangle = Tk_n^2\delta_{nm}\delta(t-t'),$$

where T is a positive constant. In the projection operator formalism, the equation corresponding to (3.5) is derived from (2.11) using the Markov approximation as follows:

$$\frac{d\hat{u}_n(t)}{dt} = -\int_0^\infty \Gamma_n(s)ds\hat{u}_n(t) + r_n(t). \tag{3.6}$$

A comparison between (3.6) and the Fourier form of (3.5) indicates that the memory term in (3.6) corresponds to both the linear term $\nu_T\tilde{u}_{xx}$ and the nonlinear term $\tilde{u}\tilde{u}_x$ in (3.5). As

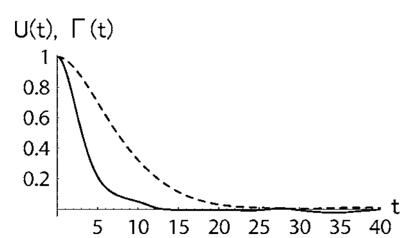

FIG. 9. Normalized time correlation functions for the short wave mode $n=55$: _ _ _, $U_{55}(t)/U_{55}$; ____, $\Gamma_{55}(t)/\Gamma_{55}$.

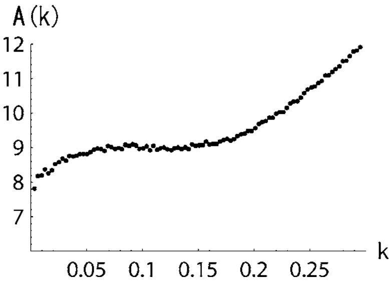

FIG. 10. $A(k_n)$ as a function of k_n.

a result of the comparison, we can obtain the relation

$$\nu_T \neq \frac{1}{k_n^2}\int_0^\infty \Gamma_n(s)ds.$$

Now, we evaluate the eddy viscosity from the memory term. The coefficient of the memory term is replaced by

$$\int_0^\infty \Gamma_n(s)ds = A(k_n)k_n^2 - Bk_n^3, \tag{3.7}$$

where $A(k_n)$ is a function of k_n and B is a constant. Replacement (3.7) is exact because $A(k_n)$ is a function of k_n. If $A(k_n)$ is independent of k_n, it is possible to interpret (3.7) as implying that $A(k_n)$ is the eddy viscosity ν_T and Bk_n^3 corresponds to the nonlinear term. Because the memory function $\Gamma_n(t)$ is exactly evaluated through a numerical simulation (see Appendix C), we can obtain $A(k_n)$ as a function of k_n if the value of B is given. Here, the value of B is decided in order to maintain $A(k_n)$ as constant as possible. Figure 10 shows the k_n dependence of $A(k_n)$ for $B=35$. Because $A(k_n)$ remains almost constant for $0.07 < k_n < 0.17$, it is considered to be the eddy viscosity for small values of k_n and its value is 9.0, which is consistent with three former results: $7.3 < \nu_T < 9.9$ by Zaleski [17], $\nu_T \approx 10.5$ by Sneppen *et al.* [18], and $\nu_T \approx 10$ by Sakaguchi [19]. It is important to note that the four methods, including the present one, to evaluate the eddy viscosity are completely different.

The figure also shows that $A(k_n)$ deviates from the constant value near 9.0 for $k_n < 0.07$ and $k_n > 0.17$. Because the concept of the eddy viscosity loses its meaning for larger values of k_n and furthermore, the nonlinear term needs more higher-order terms in k_n such as k_n^4, $A(k_n)$ deviates from the constant. However, there is no reason for $A(k_n)$ to deviate from the constant value for smaller values of k_n, because the concept of the eddy viscosity should be most suitable and

$$\nu_T = \lim_{k_n \to 0} \frac{1}{k_n^2}\int_0^\infty \Gamma_n(s)ds.$$

IV. CONCLUDING REMARKS AND DISCUSSION

By investigating the validity of the essential assumption that the chaotic motion is successfully divided into a coherent motion and a fluctuating one with the projection operator method, we have found that the assumption is reasonable for both long wave modes and short wave modes. It is natural that the essential assumption is valid for long wave modes because the long wave modes are separated from the dominant mode of the chaotic motion; further, the Markov approximation is satisfactory for long wave modes because of their separation from the dominant mode. Surprisingly, the assumption is suitable even for short wave modes such as $n=55$. The short wave modes include the dominant mode of $n=55$ and thereby, the separation of scales is not realized. This fact suggests that the projection operator method may also be useful for other systems without the separation of scales, such as Navier-Stokes turbulence.

The Markov approximation is suitable only for the long wave modes and therefore, it is expected to be most useful for the mean values, which are the limit of the long wave modes. Because the characteristic time scale of the mean value is much larger than that of the fluctuating motion, the Markov approximation is satisfactory for the evaluation of the mean values with the projection operator method [10,11].

We have shown that the memory term includes motion that is related to the eddy viscosity for smaller values of k_n, and the concept of eddy viscosity loses its meaning for $k_n > 0.17$. The value we have obtained, $\nu_T=9.0$, is consistent with the former estimates using other methods.

APPENDIX A: DERIVATION OF THE BURGERS EQUATION

We derive the Burgers equation from the KS equation. Taking $\langle u\rangle$ and $\tilde{u}$ to represent the ensemble average and fluctuating component of motion, respectively, we obtain the relation

$$\langle u\rangle_t + \langle u\rangle\langle u\rangle_x + \langle\tilde{u}\rangle\langle\tilde{u}\rangle_x + \langle u\rangle_{xx} + \langle u\rangle_{xxxx} = 0, \tag{A1}$$

from (2.1). Because the spatial change of the ensemble average is very slow, we obtain

$$|\langle u\rangle_{xx}| \gg |\langle u\rangle_{xxxx}|. \tag{A2}$$

We introduce a simple turbulence model that expresses the relation between the fluctuating component and ensemble average as follows:

$$\langle\tilde{u}\rangle\langle\tilde{u}\rangle = -\nu_E\langle u\rangle_x, \tag{A3}$$

where ν_E is the eddy viscosity in turbulent flows [14]. Using (A2) and (A3), we obtain the Burgers equation

$$\langle u\rangle_t + \langle u\rangle\langle u\rangle_x - \nu_T\langle u\rangle_{xx} = 0,$$

from (A1), where $\nu_T=\nu_E-1$. It is important to note that the difference between ν_T and ν_E corresponds to molecular viscosity in fluids, and it is not negligible in the KS equation.

APPENDIX B: HOMOGENEITY, STEADINESS, AND PARITY INVARIANCE

We expect statistical homogeneity, represented by

$$\langle\hat{u}_n(t)\hat{u}_m^*(s)\rangle = \langle\hat{u}_n(t)\hat{u}_n^*(s)\rangle\delta_{nm},$$

and

$$\langle \hat{u}_n(t)\hat{u}_m(t)\hat{u}_l(s)\rangle = 0, \quad \text{unless } n+m+l=0,$$

for a sufficiently large period, because the KS equation (2.1) is invariant under a spatial translation. Statistical homogeneity yields the following relations:

$$\Omega_{nm}(t) = \Omega_n(t)\delta_{nm}, \quad \Omega_n(t) \equiv \Omega_{nn}(t), \tag{B1}$$

and

$$\Gamma_{nm}(t) = \Gamma_n(t)\delta_{nm}, \quad \Gamma_n(t) \equiv \Gamma_{nn}(t), \tag{B2}$$

where δ_{nm} denotes the Kronecker delta and there is no summation over repeated subscripts.

We can also expect statistical steadiness, expressed by

$$\langle \hat{u}_n(t)\hat{u}_m^*(s)\rangle = \langle \hat{u}_n(t-s)\hat{u}_m^*\rangle,$$

and

$$\frac{d\langle|\hat{u}_n(t)|^2\rangle}{dt} = 0, \tag{B3}$$

for a sufficiently large time, because the KS equation is invariant with respect to time translation.

Parity invariance is also consistent with the KS equation. In other words, the KS equation is invariant under a transformation that reverses the signs of both x and u, which corresponds to $\hat{u}_n \to -\hat{u}_{-n}$ in Fourier space. For example, we expect the following statistical property:

$$\langle \hat{u}_j\hat{u}_{-n-j}\hat{u}_{-n}^*\rangle = -\langle \hat{u}_{-j}\hat{u}_{n+j}\hat{u}_n^*\rangle. \tag{B4}$$

Using (2.3), (2.5), (2.8), (2.14), and (B1), we obtain

$$\Omega_n = -\frac{i}{2LU_n}\sum_{j=-N}^{N} k_j\langle \hat{u}_j\hat{u}_{n-j}\hat{u}_n^*\rangle,$$

and

$$\Omega_{-n} = -\frac{i}{2LU_n}\sum_{j=-N}^{N} k_j\langle \hat{u}_j\hat{u}_{-n-j}\hat{u}_{-n}^*\rangle. \tag{B5}$$

Replacing j by $-j$ in (B5) and using the relation $\hat{u}_n=\hat{u}_{-n}^*$, we obtain

$$\Omega_{-n} = \frac{i}{2LU_n}\sum_{j=-N}^{N} k_j\langle \hat{u}_j^*\hat{u}_{n-j}^*\hat{u}_n\rangle = \Omega_n^*. \tag{B6}$$

On the other hand, by replacing j with $-j$ in (B5) and using (B4), we find

$$\Omega_{-n} = -\frac{i}{2LU_n}\sum_{j=-N}^{N} k_j\langle \hat{u}_j\hat{u}_{n-j}\hat{u}_n^*\rangle = \Omega_n. \tag{B7}$$

Then, from (B6) and (B7), we have

$$\text{Im}\,\Omega_n = 0. \tag{B8}$$

The steadiness condition (B3) and the basic equation (2.2) yield

$$2L_n\langle \hat{u}_n\hat{u}_n^*\rangle + \langle \hat{N}_n\hat{u}_n^*\rangle + \langle \hat{N}_n^*\hat{u}_n\rangle = 0. \tag{B9}$$

Hence, (B9) can be rewritten

$$L_n + \text{Re}\,\Omega_n = 0, \tag{B10}$$

using the homogeneous version of (2.8). Therefore, from (B8) and (B10), we can obtain the relation

$$L_n + \Omega_n = 0.$$

By using (2.6) and performing the partial integral, we obtain the relation

$$\left\langle \frac{d\hat{u}_n(t)}{dt}\hat{u}_n^*\right\rangle + \left\langle \hat{u}_n(t)\frac{d\hat{u}_n^*}{dt}\right\rangle = 0,$$

which is transformed into

$$2L_n\langle \hat{u}_n(t)\hat{u}_n^*\rangle + \langle N_n(t)\hat{u}_n^*\rangle + \langle \hat{u}_n(t)N_n^*\rangle = 0. \tag{B11}$$

Equation (B11) is an extension of (B9).

APPENDIX C: NUMERICAL EVALUATION OF MEMORY FUNCTION $\Gamma_n(t)$

From (2.2) and (2.11), we obtain the relation

$$\langle f_n(t)f_n^*\rangle = -\int_0^t \Gamma_n(s)\langle \hat{u}_n(t-s)r_n^*\rangle ds + \langle r_n(t)r_n^*\rangle, \tag{C1}$$

using $f_n^*=r_n^*$, and $\langle \hat{u}_n(t)r_n^*\rangle$ is transformed as follows:

$$\begin{aligned}\langle \hat{u}_n(t)r_n^*\rangle &= \langle \hat{u}_n(t)\mathcal{Q}N_n^*\rangle \\ &= \langle \hat{u}_n(t)N_n^*\rangle - \langle \hat{u}_n(t)\mathcal{P}N_n^*\rangle \\ &= -2L_n\langle \hat{u}_n(t)\hat{u}_n^*\rangle - \langle N_n(t)\hat{u}_n^*\rangle - \Omega_n^*\langle \hat{u}_n(t)\hat{u}_n^*\rangle \\ &= -\langle f_n(t)\hat{u}_n^*\rangle,\end{aligned} \tag{C2}$$

where we have used (2.2), (2.4), (2.8), (2.9), and (B11). It is important to note that (C2) is derived under the assumptions of statistical homogeneity, steadiness, and parity invariance. Relations (C1) and (C2) yield

$$\Gamma_n(t) = \frac{1}{U_n}F_n(t) - \frac{1}{U_n}\int_0^t \Gamma_n(s)\langle f_n(t-s)\hat{u}_n^*\rangle ds. \tag{C3}$$

We can obtain the memory function $\Gamma_n(t)$ from (C3) by using an iteration method with an initial estimate $\Gamma_n(t)=F_n(t)/U_n$ because U_n, $F_n(t)$, and $\langle f_n(t-s)\hat{u}_n^*\rangle$ have already been numerically evaluated.

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